BRILLIANT POINTS AND LOCI OF BRILLIANT POINTS.*

By W. H. Roever.

1. Introduction. When a ray of light meets a polished surface it is reflected according to the following laws:

The incident ray, the reflected ray, and the normal to the reflecting surface at the point of reflection lie in one plane. This plane is called the plane of reflection.

In the plane of reflection, the incident ray and the reflected ray lie on opposite sides of the normal and on the same side of the tangent.

The incident ray and the reflected ray make equal angles with the normal.

These laws are inferred from experiments in which pencils of light are reflected from polished planes.

The points of reflection of those reflected rays which pass through the pupil of an observer's eye are, for the observer, luminous points, and are called *brilliant points.*† The points of reflection of those reflected rays which, when produced backward, pierce the pupil of an observer's eye are *not* luminous, and are sometimes called *virtual* brilliant points. Virtual brilliant points present themselves in the analytic treatment when certain equations are freed from radicals.

Tubular surfaces are envelopes of spheres of constant radii whose centres are situated in a given space curve which is called the axis of the tube. As the radius of a tubular surface approaches zero, its brilliant points approach points in the axis. The axis is the given space curve, and this fact suggests the definition of a brilliant point of a space curve.

It is evident that the position of a brilliant point depends upon the position of the observer's eye, the position and nature of the source of light, and the position of the reflecting surface. Hence when the surface, source or eye moves, or when the surface changes its shape or size, the brilliant points move. The locus of a brilliant point may be a curve, a surface or a region of more

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* Presented to the American Mathematical Society at its meeting, 22 February, 1902. A communication on this subject has been made to The Academy of Science of St. Louis; *cf.* *Trans. Acad. Sci. of St. Louis*, vol. x, No 11 (1900), p. lxii.

† Alhazen (A. D. 987–1038) determined geometrically the brilliant points of a concave mirror when the source and the recipient are given. *Cf.* Ball, *A Short History of Mathematics*, p. 167.

dimensions. When it is a curve, it is called a *brilliant curve*, and when a surface, a *brilliant surface*. Owing to the fact that the impression of an image on the retina of the eye remains for some time after the object which has produced it has disappeared or become displaced, it is possible for us to perceive the brilliant curve when the illuminated polished surface is rapidly moved.

A striking example of this fact is presented by an illuminated polished rod which is rapidly rotated. When the rod is of small radius and the axis of rotation intersects and is perpendicular to the rod, the brilliant curve lies in a plane. Fig. 3 illustrates this case. The plane of the paper represents the plane of rotation, P_1' and P_2' are the projections of the source of light and of the eye respectively on this plane, and z_1 and z_2 are the distances of the light and of the eye above the plane. The heavy full line represents the visible portion of the brilliant curve. This curve may be seen in a carriage wheel. In this case the source of light is either the sun or a street lamp. Owing to the great distances of the source of light and the eye of the observer from the hub of the wheel (great in comparison with the diameter of the wheel), the brilliant curve looks like a straight line passing through the hub, and on account of the motion of the carriage this line continually changes its position.

Closely packed assemblages of brilliant points look like continuous regions of light when the consecutive brilliant points are so near each other that the eye can not separate them. An example of this is presented by a circular saw which has been polished with emery in a lathe. The consecutive scratches made by the particles of emery are so close that their corresponding brilliant points are too close to be separated by the eye. Each scratch may be considered as being the special position of a variable scratch, and hence the isolated brilliant points are points of a brilliant curve. This curve is illustrated by Fig. 2 and the photographs. In Fig. 2 the plane of the paper below the line AB represents the plane of the saw, P_1' and P_2' are the projections of the source of light and of the eye respectively on this plane, and z_1 and z_2 are the distances of the light and of the eye above the plane. The heavy full line represents the visible portion of the brilliant curve. The photographs also show this curve. In the latter case the source of light is an electric arc and the eye is replaced by the optical centre of the lens of the camera. It will be observed, in the photographs, that near the centre of the saw there is a discontinuity in the brilliant curve. This is due to the fact that, at a small distance from the centre, the concentric circular scratches are replaced by spiral scratches, due to a different mode of polishing.

The most general problem to be discussed in this paper is the following:— *Required to find the locus of the brilliant points of a two parameter family of space curves when the source of light and the eye of an observer are in given fixed positions.* An example under this case is presented by the rapid rotation of the saw above referred to, about an axis in its plane. Another example is furnished by a family of parallel polished wires, as for instance the wires strung on the arms of a telegraph pole.

In what has been said above it has been tacitly assumed that the source of light is a point and that all the rays reflected at the brilliant points belong to a family of right lines each member of which passes through the same point, namely, the pupil of the observer's eye. It might be required to find those points of reflection for which the reflected rays belong to a family of right lines each member of which is normal to a given surface. This given surface would be called the *recipient.* The eye would then be said to be a *point recipient.* A surface to which all the incident rays are normal would be called the *source.* When all the incident rays emanate from a single point, this point would be said to be a *point source.*

In the present paper I shall confine myself to the brilliant points of curves with respect to a point source and a point recipient.

2. Definition of Brilliant Points. Given the space curve

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0,$$

a family of right lines emanating from the point P_1, (x_1, y_1, z_1) called the *source*, a family of right lines emanating from the point P_2, (x_2, y_2, z_2) called the *recipient.* The point P_0, (x_0, y_0, z_0) is said to be a *brilliant point of the space curve $F_1 = 0$, $F_2 = 0$ with respect to the points P_1 and P_2* when the following conditions are fulfilled.

1. The point P_0 is a point of the curve $F_1 = 0$, $F_2 = 0$;
2. The right line P_0P_1, considered as a whole, and the right line P_0P_2, considered as a whole, make equal angles with the line of intersection of their plane with the plane normal to the curve $F_1 = 0$, $F_2 = 0$ at P_0. If the two planes coincide, P_0 shall still be considered as fulfilling this condition.

This is the most natural definition of the brilliant point of a space curve as suggested by that of a surface. Condition 2 is co-extensive with the following condition.

3. The right line $P_0 P_1$, considered as a whole, and the right line $P_0 P_2$, considered as a whole, make equal angles with the right line which is tangent to the curve $F_1 = 0$, $F_2 = 0$ at P_0.

Fig. 1 enables us to see geometrically that Conditions 2 and 3 are coextensive. In the figure, P_0 is the centre of a sphere which is pierced in the points A, A by the tangent at P_0 to the space curve $F_1 = 0$, $F_2 = 0$, in the point B_1 by the right line $P_0 P_1$, in the point B_2 by the right line $P_0 P_2$, and in the point C by the line of intersection of the normal plane and the plane of the lines $P_0 P_1$, $P_0 P_2$. The great circle through C which has A, A for poles is the intersection of the sphere by the plane which is normal to the

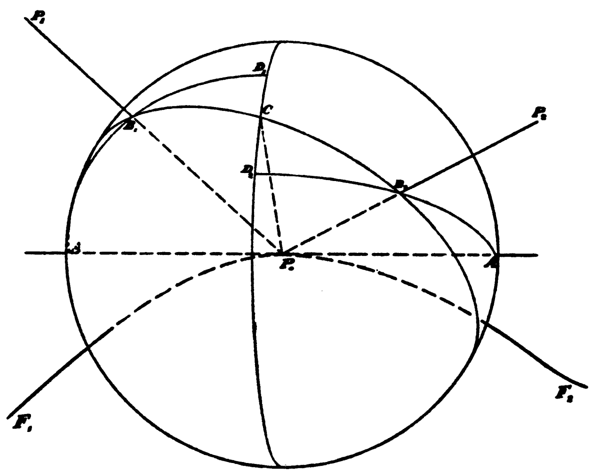

FIG. 1.

curve $F_1 = 0$, $F_2 = 0$ at P_0. The great circles AB_1 and AB_2 cut the normal plane in D_1 and D_2 respectively. In the right spherical triangles $B_1 D_1 C$ and $B_2 D_2 C$, angle $D_1 C B_1 =$ angle $D_2 C B_2$. By Condition 2, side $CB_1 =$ side CB_2. Hence the two triangles are equal and arc $AB_1 =$ arc AB_2. But this is Condition 3. If, on the other hand, we have given arc $AB_1 =$ arc AB_2, it follows that arc $CB_1 =$ arc CB_2, and this is Condition 2.

Classification of Brilliant Points. Those brilliant points P_0 for which the segments of right lines $P_0 P_1$ and $P_0 P_2$ lie on opposite knappes of a cone of revolution whose axis is the tangent to the space curve at P_0 and whose vertex is P_0, shall be called *actual brilliant points*, and those for which these segments lie on the same knappe shall be called *virtual brilliant points.*

When the semi-angle of this cone is a right angle, an ambiguity arises. In this case P_0 shall be called an actual brilliant point, except when it lies between P_1 and P_2 on the right line P_1P_2. In this exceptional case, it shall be called a virtual brilliant point.

If the curve represented by the equations $F_1 = 0$, $F_2 = 0$ be the axis of a polished wire of small radius, and if one of the points P_1, P_2 be replaced by a source of light and the other by the eye of an observer, then the observer will see the actual brilliant points as luminous points.

3. Analytical Conditions. The deduction of the following conditions will be given in a paper which the author intends to publish in the near future.

The necessary and sufficient condition that the point* $P, (x, y, z)$ *shall be a brilliant point of the space curve* $F_1(x, y, z) = 0$, $F_2(x, y, z) = 0$, *with respect to the two points* $P_1, (x_1, y_1, z_1)$ *and* $P_2, (x_2, y_2, z_2)$ *is*

$$\left.\begin{aligned} F_1(x, y, z) &= 0 \\ F_2(x, y, z) &= 0 \end{aligned}\right\} \qquad (a)$$

$$\begin{aligned} A_2\Big\{ & B_1\Big[\frac{\partial F_1}{\partial x}(x - x_2) + \frac{\partial F_1}{\partial y}(y - y_2) + \frac{\partial F_1}{\partial z}(z - z_2)\Big] \\ & + B_2\Big[\frac{\partial F_1}{\partial x}(x - x_1) + \frac{\partial F_1}{\partial y}(y - y_1) + \frac{\partial F_1}{\partial z}(z - z_1)\Big]\Big\} \\ - A_1\Big\{ & B_1\Big[\frac{\partial F_2}{\partial x}(x - x_2) + \frac{\partial F_2}{\partial y}(y - y_2) + \frac{\partial F_2}{\partial z}(z - z_2)\Big] \\ & + B_2\Big[\frac{\partial F_2}{\partial x}(x - x_1) + \frac{\partial F_2}{\partial y}(y - y_1) + \frac{\partial F_2}{\partial z}(z - z_1)\Big]\Big\} \\ & = 0, \qquad (b) \end{aligned}$$

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* This condition is sufficient, provided that there are no points at which the surfaces $F_1 = 0$, $F_2 = 0$ are tangent and hence

$$\begin{vmatrix} \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial F_1}{\partial z} & \frac{\partial F_1}{\partial x} \\ \frac{\partial F_2}{\partial z} & \frac{\partial F_2}{\partial x} \end{vmatrix} = 0, \quad \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = 0;$$

and no points at which either $\frac{\partial F_1}{\partial x} = 0$, $\frac{\partial F_1}{\partial y} = 0$, $\frac{\partial F_1}{\partial z} = 0$ or $\frac{\partial F_2}{\partial x} = 0$, $\frac{\partial F_2}{\partial y} = 0$, $\frac{\partial F_2}{\partial z} = 0$.

where

$$A_i = \begin{vmatrix} \frac{\partial F_i}{\partial x} & \frac{\partial F_i}{\partial y} & \frac{\partial F_i}{\partial z} \\ x - x_1 & y - y_1 & z - z_1 \\ x - x_2 & y - y_2 & z - z_2 \end{vmatrix}, \qquad B_i = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} & \frac{\partial F_1}{\partial z} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} & \frac{\partial F_2}{\partial z} \\ x - x_i & y - y_i & z - z_i \end{vmatrix}, \quad i = 1, 2.$$

By putting, in equations (a), (b),

$$F_1(x, y, z) = F(x, y) = 0, \qquad F_2(x, y, z) = z = 0,$$

and hence

$$\frac{\partial F_2}{\partial x} = 0, \qquad \frac{\partial F_2}{\partial y} = 0, \qquad \frac{\partial F_2}{\partial z} = 1,$$

$$\frac{\partial F_1}{\partial x} = \frac{\partial F}{\partial x}, \qquad \frac{\partial F_1}{\partial y} = \frac{\partial F}{\partial y}, \qquad \frac{\partial F_1}{\partial z} = 0,$$

in which $F(x, y)$, $\frac{\partial F}{\partial x}$, $\frac{\partial F}{\partial y}$ are functions of the variables x and y only, we obtain the following theorem:

The necessary and sufficient condition that the point $P, (x, y)$ shall be a brilliant point of the plane curve $F(x, y) = 0$, with respect to the two points $P_1, (x_1, y_1, z_1)$ and $P_2, (x_2, y_2, z_2)$, which are not in the plane of the curve is

$$F(x, y) = 0 \tag{a'}$$

$$\begin{aligned} \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \Bigg\{ & \left[\frac{\partial F}{\partial x}(x - x_1) + \frac{\partial F}{\partial y}(y - y_1)\right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix} \\ & + \left[\frac{\partial F}{\partial x}(x - x_2) + \frac{\partial F}{\partial y}(y - y_2)\right] \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix} \Bigg\} \\ & + z_1^2 \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix}^2 - z_2^2 \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix}^2 = 0 \end{aligned} \tag{b'}$$

Let us denote by a_1 and a_2 the absolute values of z_1 and z_2 respectively

and by P_1' and P_2' the projections of the points P_1 and P_2 on the X, Y plane. The condition just stated is the same for the four cases: $(z_1 = a_1,\ z_2 = a_2)$, $(z_1 = -a_1,\ z_2 = -a_2)$, $(z_1 = -a_1,\ z_2 = a_2)$, $(z_1 = a_1,\ z_2 = -a_2)$. According as we have one of the second two, or one of the first two of these cases, the right line $P_1 P_2$ cuts the right line $P_1' P_2'$ between or not between P_1' and P_2'. Let us denote by P_i the point between P_1' and P_2' and by P_j the other point. The four points P_j, P_1', P_i, P_2' form a harmonic range. The coordinates of P_i and P_j are respectively

$$\left(\frac{x_1 + \dfrac{a_1}{a_2} x_2}{1 + \dfrac{a_1}{a_2}},\ \frac{y_1 + \dfrac{a_1}{a_2} y_2}{1 + \dfrac{a_1}{a_2}}\right),\ \left(\frac{x_1 - \dfrac{a_1}{a_2} x_2}{1 - \dfrac{a_1}{a_2}},\ \frac{y_1 - \dfrac{a_1}{a_2} y_2}{1 - \dfrac{a_1}{a_2'}}\right).$$

Both P_i and P_j satisfy equation (b'), and hence if the curve (a') passes through them, they are brilliant points.

By putting, in equation (b'), $z_1 = 0$ and $z_2 = 0$, we obtain the following theorem:

The necessary and sufficient condition that the point $P, (x, y)$ shall be a brilliant point of the plane curve $F(x, y) = 0$, with respect to the two points $P_1, (x_1, y_1)$ and $P_2, (x_2, y_2)$, which are in the plane of the curve is

$$F(x, y) = 0 \qquad (a'')$$

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \left\{ \left[\frac{\partial F}{\partial x}(x - x_1) + \frac{\partial F}{\partial y}(y - y_1)\right] \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ x - x_2 & y - y_2 \end{vmatrix} + \left[\frac{\partial F}{\partial x}(x - x_2) + \frac{\partial F}{\partial y}(y - y_2)\right] \begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ x - x_1 & y - y_1 \end{vmatrix} \right\} = 0 \qquad (b'')$$

The first factor of the left hand member of equation (b'') vanishes for all points of the right line $P_1 P_2$. With the exception of P_1 and P_2, the second factor vanishes for *no* point of this right line, except when the line is normal or tangent to the curve $F = 0$. To show this, substitute for x and y the coordinates of any point of the right line $P_1 P_2$

$$x = \frac{x_1 + k x_2}{1 + k}, \qquad y = \frac{y_1 + k y_2}{1 + k}.$$

The second factor of the left hand member of (b'') then becomes

$$-\frac{2k}{(1+k)^2}\left[\frac{\partial F}{\partial x}(x_1-x_2)+\frac{\partial F}{\partial y}(y_1-y_2)\right]\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ x_1-x_2 & y_1-y_2 \end{vmatrix}.$$

This expression vanishes when $k=0$ and when $k=\infty$. These are the coordinates of P_1 and P_2. For other values of k the expression cannot vanish unless

either
$$\frac{\partial F}{\partial x}(x_1-x_2)+\frac{\partial F}{\partial y}(y_1-y_2)=0 \tag{1}$$

or
$$\begin{vmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ x_1-x_2 & y_1-y_2 \end{vmatrix}=0. \tag{2}$$

If the right line P_1P_2 and the curve $F(x, y)=0$ meet, equation (1) is the necessary and sufficient condition that P_1P_2 shall be tangent to $F=0$, and equation (2) is the necessary and sufficient condition that P_1P_2 shall be normal to $F=0$.

4. Loci of Brilliant Points. Suppose we have a family of space curves given by the equations

$$\Phi_1(x, y, z, p_1)=0, \qquad \Phi_2(x, y, z, p_2)=0. \tag{a}$$

in which p_1 and p_2 are two independent parameters. If, after substituting Φ_1 and Φ_2 for F_1 and F_2 in equations (a), (b), we eliminate p_1 and p_2 between these three equations, there results an equation, independent of p_1 and p_2, which is satisfied by the brilliant points of every member of the two parameter family of space curves (a). This equation represents the surface which is the locus of all the brilliant points, *i. e.* the *brilliant surface*, of the given family of space curves with respect to the fixed points P_1 and P_2. When the functions Φ_1 and Φ_2 are of the form

$$\Phi_1(x, y, z, p_1)=F_1(x, y, z)-p_1, \qquad \Phi_2(x, y, z, p_2)=F_2(x, y, z)-p_2,$$

no elimination is necessary, and equation (b), as it stands, represents the brilliant surface. Similarly, if we have a family of plane curves given by the equation

$$\Phi(x, y, p)=0, \tag{a'}$$

and if, after substituting Φ for F in equations (a'), (b'), we eliminate p between these equations, we obtain the equation of the *brilliant curve* of the given family of plane curves (a') with respect to the fixed points P_1 and P_2, which are not in the plane of the curves. When the function Φ is of the form

$$\Phi(x, y, p) = F(x, y) - p,$$

no elimination is necessary, and equation (b'), as it stands, represents the brilliant curve. When both P_1 and P_2 lie in the plane of the family of curves (a'), we obtain the equation of the brilliant curve by eliminating p between equations (a''), (b'') after substituting Φ for F. In this case the brilliant curve consists of two distinct curves, one of which is the right line connecting P_1 and P_2.

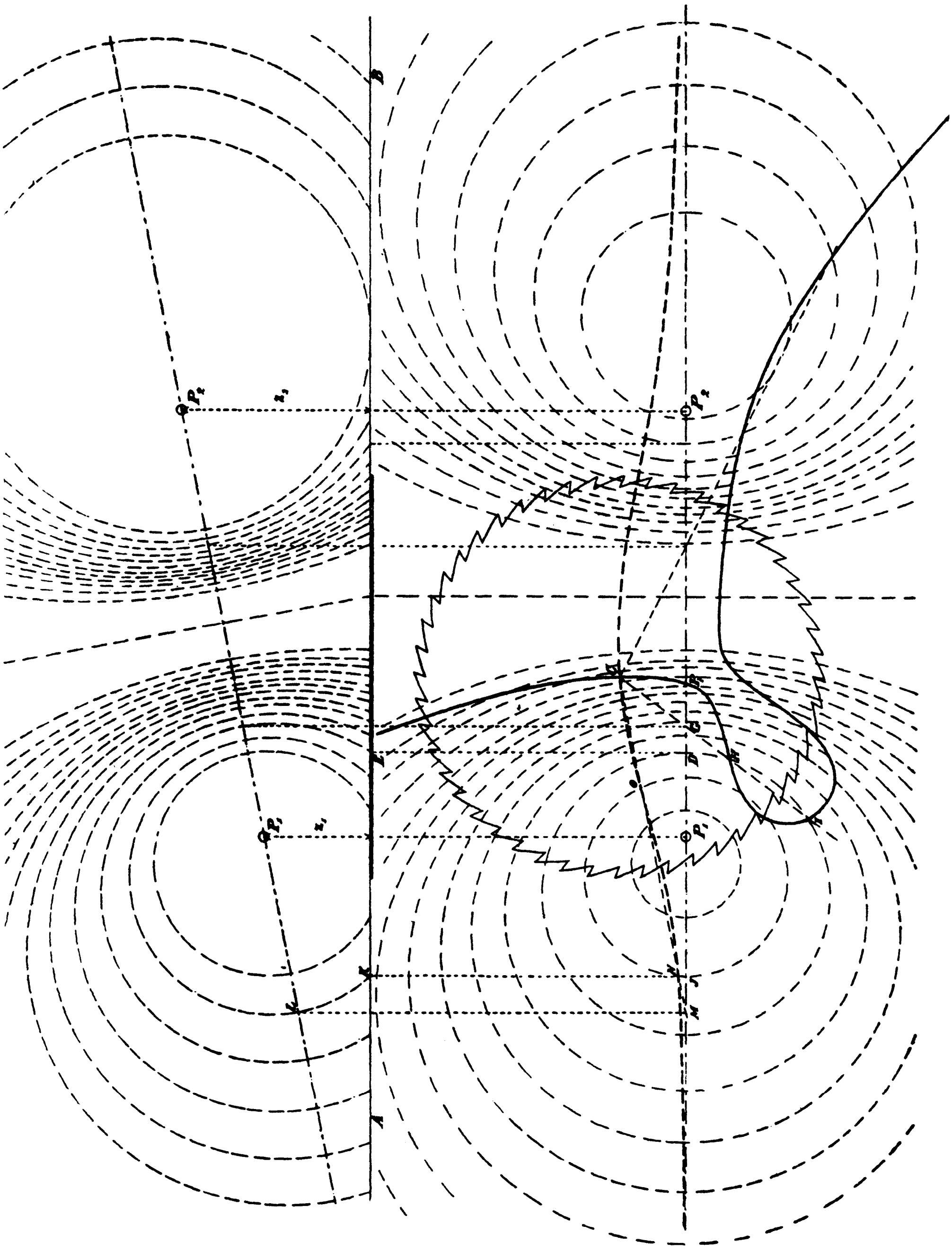
THEOREM. *When the source and the recipient are in the plane of a family of curves, the brilliant curve of this family and that of its orthogonal trajectories are identical. The portion of the brilliant curve, with the exception of the right line $P_1 P_2$, which for the one family is the locus of the actual brilliant points, is for the other family the locus of the virtual brilliant points, and vice versa.*

5. Applications. (*a*) *The Saw Curve.* Let it be required to find the equation of the "saw curve" referred to in the introduction. The curve of light seen by an observer and shown in the accompanying photographs is that portion of the brilliant curve which is the locus of the actual brilliant points. In Fig. 2 the locus of the actual brilliant points is represented by a heavy full line and that of the virtual brilliant points by a heavy dashed line. The scratches caused by the particles of emery are concentric circles whose common centre is the centre of the saw, and hence our family of curves is represented by the equation

$$x^2 + y^2 - a^2 = 0, \tag{3}$$

in which a is the variable parameter. $P_1, (x_1, y_1, z_1)$ is the source and P_2, (x_2, y_2, z_2) is the recipient. The derivatives of the left hand member of equation (3) are

$$\frac{\partial F}{\partial x} = 2x, \qquad \frac{\partial F}{\partial y} = 2y.$$



Substituting these in (b') we obtain

$$\Big[(x_2 y - y_2 x) + (y_1 x - x_1 y) + (x_1 y_2 - x_2 y_1)\Big]\Big[\{x(x - x_1) + y(y - y_1)\}$$
$$\{x_2 y - y_2 x\} + \{x(x - x_2) + y(y - y_2)\}\{x_1 y - y_1 x\}\Big]$$
$$+ z_1^2(x_2 y - y_2 x)^2 - z_2^2(x_1 y - y_1 x)^2 = 0. \tag{4}$$

This equation may also be written as follows:

$$\Big[(x_2 - x_1)y - (y_2 - y_1)x + x_1 y_2 - x_2 y_1\Big]\Big[(x_2 + x_1)(y^3 + x^2 y)$$
$$- (y_2 + y_1)(x^3 + y^2 x) + (x_1 y_2 + y_1 x_2)(x^2 - y^2) + 2(y_1 y_2 - x_1 x_2)xy\Big]$$
$$- 2(z_1^2 x_2 y_2 - z_2^2 x_1 y_1)xy + (z_1^2 x_2^2 - z_2^2 x_1^2)y^2 + (z_1^2 y_2^2 - z_2^2 y_1^2)x^2 = 0. \tag{5}$$

The result may be stated thus: *The brilliant curve of a family of concentric circles, when the source and recipient are not in the plane of the family, is a curve of the fourth degree.* This curve passes through the common centre of the circles (3), and also through the points P_i and P_j, whose coordinates have been given in §3.

When the right line $P_1'P_2'$, which is the projection of P_1P_2 on the plane of the circles, passes through the centre of the circles, $\dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$. For this relation, equation (4) becomes

$$(cy - x)^2\Big[(y_2^2 - y_1^2)(x^2 + y^2) - (y_2 - y_1)(x_1 y_2 + x_2 y_1)x - 2(y_1 y_2)(y_2 - y_1)y$$
$$+ z_1^2 y_2^2 - z_2^2 y_1^2\Big] = 0, \tag{6}$$

in which $c = \dfrac{x_1}{y_1} = \dfrac{x_2}{y_2}$. The first factor of equation (6), when set equal to zero, represents two coincident right lines which pass through the origin and contain P_1' and P_2'. The second factor, when set equal to zero, represents the circle cut by the (X, Y)-plane from the sphere whose equation is

$$\frac{(x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2}{(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2} = k^2, \tag{7}$$

in which $k = \dfrac{x_1}{x_2} = \dfrac{y_1}{y_2}$. This sphere cuts the right line P_1P_2 in points whose

coordinates are

$$\frac{x_1 \pm kx_2}{1 \pm k}, \qquad \frac{y_1 \pm ky_2}{1 \pm k}, \qquad \frac{z_1 \pm kz_2}{1 \pm k}.$$

On account of the value of k, the first two coordinates of one of these points are zero. The result of putting $\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} = 0$ can be expressed as follows:

When the axis of the saw (i. e. the right line perpendicular to the plane of the saw at its centre) intersects the right line $P_1 P_2$, which connects the source of light, P_1, and the recipient, an observer's eye, P_2, the brilliant curve consists of a straight line and a circle. The straight line passes through the centre of the saw and contains the projections of P_1 and P_2. The circle is the intersection of the plane of the saw by the sphere which has P_1 and P_2 as conjugate points, and passes through the intersection of the axis of the saw and the right line $P_1 P_2$.

When the axis of the saw intersects the right line $P_1 P_2$ in the point P_3 half way between P_1 and P_2, the sphere is replaced by the plane which is perpendicular to $P_1 P_2$ at P_3, and in this case the brilliant curve consists of two right lines which are perpendicular to each other and one of which passes through the centre of the saw. When in this special case the right line $P_1 P_2$ is parallel to the plane of the saw, the two right lines cross each other perpendicularly at the centre of the saw.

The photographs show some of the special cases.

When $z_1 = 0$ and $z_2 = 0$, the curve represented by equation (5) degenerates into the curves represented by the equations*

$$(x_2 - x_1)y - (y_2 - y_1)x + x_1 y_2 - x_2 y_1 = 0 \tag{8}$$

and

$$(x_2 + x_1)(y^3 + x^2 y) - (y_2 + y_1)(x^3 + y^2 x) + (x_1 y_2 + y_1 x_2)(x^2 - y^2) + 2(y_1 y_2 - x_1 x_2)xy = 0. \tag{9}$$

Equations (8), (9) might have been gotten directly from (b''). It is, of course, a physical impossibility to see the curve represented by equation (9).

(*b*) *The Carriage Wheel Curve.* Let it be required to find the equation of the "carriage wheel curve" referred to in the introduction. In Fig. 3

* Equation (9) is the equation obtained by Lieutenant Hamilton, ANNALS OF MATHEMATICS, ser. 2, vol. 2 (1900/01), p. 97. A geometrical construction for this curve is given by Eagles, *Constructive Geometry of Plane Curves*, p. 333.

the locus of the actual brilliant points is represented by a heavy full line and that of the virtual brilliant points by a heavy dashed line. The spokes of the wheel being radial and of small radius, our family of curves is represented by the equation

$$y - ax = 0. \tag{10}$$

$$\frac{\partial F}{\partial x} = -a, \qquad \frac{\partial F}{\partial y} = 1.$$

Substituting these values in equation (b'), and eliminating a between the equation thus obtained and equation (10), we get

$$\Big[(x_2 y - y_2 x) + (y_1 x - x_1 y) + (x_1 y_2 - x_2 y_1)\Big](-1)\Big[\{x_1 y - y_1 x\}\{x(x - x_2)$$
$$+ y(y - y_2)\} + \{x_2 y - y_2 x\}\{x(x - x_1) + y(y - y_1)\}\Big] +$$
$$z_1^2\{x(x - x_2) + y(y - y_2)\}^2 - z_2^2\{x(x - x_1) + y(y - y_1)\}^2 = 0. \tag{11}$$

The result may be stated as follows : *The brilliant curve of a family of radiating right lines, when the source and the recipient are not in the plane of the family, is a curve of the fourth degree.* This curve passes through the *radiant*, and also through the points P_i and P_j, whose coordinates have been given in §3. The first term of equation (11) differs from that of equation (4) only in sign.

When $z_1 = 0$ and $z_2 = 0$, the curve represented by equation (11) degenerates into the curves represented by equations (8), (9). This is in accordance with the theorem of §4, since the families of curves represented by equations (3) and (10) are orthogonal trajectories.

(*c*) *A Family of Equilateral Hyperbolas.* Let it be required to find the brilliant curve of the family of plane curves

$$xy = a, \tag{12}$$

when the source and recipient are not in the plane of the curves.

$$\frac{\partial F}{\partial x} = y, \quad \frac{\partial F}{\partial y} = x.$$

Substituting these values in (b'), we get

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_1 & 1 \end{vmatrix} \Big[\{ y(x-x_1) + x(y-y_1) \} \{ y(y-y_2) - x(x-x_2) \} + \{ y(x-x_2)$$

$$+ x(y-y_2) \} \{ y(y-y_1) - x(x-x_1) \} \Big] + z_1^2 \{ y(y-y_2) - x(x-x_2) \}^2$$

$$- z_2^2 \{ y(y-y_1) - x(x-x_1) \}^2 = 0 \qquad (13)$$

6. Geometrical Constructions of the Brilliant Curve for Certain Families of Plane Curves. It is a property of the sphere that, if any point of its surface be connected by right lines with a pair of conjugate points and also with the points in which the sphere is pierced by the diameter containing the conjugate points, the latter lines bisect internally and externally the angle formed by the former.

We can make use of this property to test whether a point P of a space curve is a brilliant point with respect to a source P_1 and a recipient P_2. At P draw a plane normal to the curve and through the point P_a in which this plane is pierced by the right line $P_1 P_2$ draw a sphere having P_1 and P_2 as conjugate points. The point P is or is not a brilliant point according as this sphere does or does not pass through it. If the normal plane is parallel to the right line $P_1 P_2$, the sphere is replaced by the plane which is perpendicular to $P_1 P_2$ at the point half way between P_1 and P_2. If the normal plane contains both P_1 and P_2, the point P is by definition a brilliant point. On this principle is based a geometrical construction for finding the brilliant curve of a family of parallel curves, whose common plane we shall call the horizontal plane; with respect to a source and a recipient which are not in the horizontal plane:

To any curve, at any point, draw a normal; this line will be normal to every curve of the family. At the point in which it cuts the horizontal projection of $P_1 P_2$ draw a vertical line, and through the point in which this line cuts the right line $P_1 P_2$ draw a sphere having P_1 and P_2 as conjugate points. This sphere cuts the horizontal plane in a circle which cuts the normal in two points of the required locus.

In Fig. 2 this method is used for finding the brilliant curve of a family of concentric circles.

From the second definition of a brilliant point (see Condition 3) we

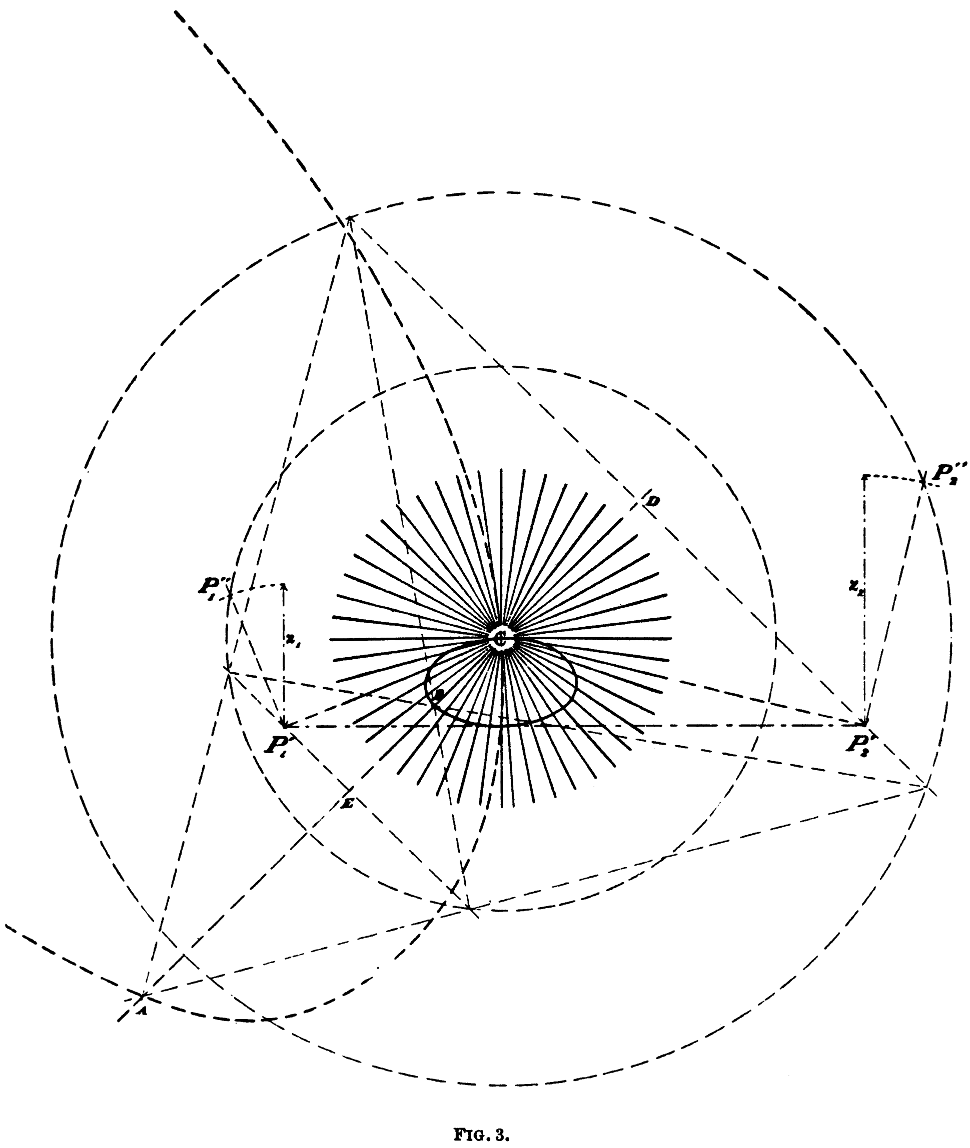


FIG. 3.

obtain the following geometrical construction for finding the brilliant points of a right line l with respect to a point source P_1 and a point recipient P_2.

About l as an axis revolve P_1 until it comes into the plane of l and P_2. There will be two revolved positions. Connect by right lines P_2 with each of the revolved positions of P_1. These right lines cut the line l in its brilliant points.

To find the locus of the brilliant points of a family of right lines, find the brilliant points of each member of the family separately.

In Fig. 3 this method is used for finding the brilliant curve of a family of radiating right lines, all of which lie in one plane, with respect to a source and recipient not in this plane.

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